A brief introduction to robust $\mathcal{H}_\infty$ control

Jean-Marc Biannic

System Control and Flight Dynamics Department

ONERA, Toulouse.

http://www.onera.fr/staff/jean-marc-biannic/
http://jm.biannic.free.fr/

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Introduction and context

Over the past thirty years, thanks to the fast development of more and more powerful computers, Robust control theory has known a large success. This field is today among the most important ones in Automatic control.

In short, robust control methods aim at providing controllers from simplified models but which still work on the real plants which are most often too complicated to be accurately described by a set of linear differential equations.

The aim of this talk is to give the main ingredients for a better understanding of this field.

Outline

Definition of robustness

In the general field of **Automatic Control**, and more precisely in **Control theory**, the notion of robustness quantifies the sensitivity of a controlled system with respect to internal or external disturbing phenomena.

External disturbances in aerospace applications

- Windshear
- Atmospheric turbulences
- Solar pressure

Examples of internal disturbances

- Parametric variations of badly known parameters
- Modelling errors (see next slide)
- Digital implementation $\Rightarrow$ *sampling time*
- Limited capacity of actuators
- Limited speed and accuracy of sensors

Why is robust control so essential ?

STABILTY & PERFORMANCE ARE ENSURED   STABILITY ? PERFORMANCE ?
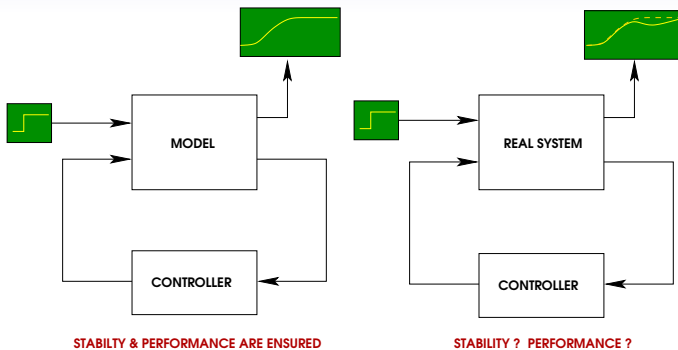
OK WITH A ROBUST CONTROLLER !

Closing the loops: what for ?

Open-loop control techniques can be used if :

- a very accurate model of the plant is available
- the plant is invertible
- external pertubations are negligible

The above conditions are rarely fullfilled. This is why it is highly preferable to use a feedback control structure which offers nice robustness properties...

OPEN−LOOP

(feed-forward control architecture)

CLOSED−LOOP
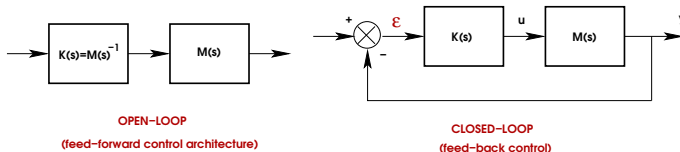
(feed-back control)

Robustness of P.I. Feedback Controllers

As an illustration of the previous slide, it is easily shown for example that the static behaviour of the following closed-loop system is neither affected by external step-like perturbations nor by gain uncertainties on the nominal plant.


P.I. Controlled system
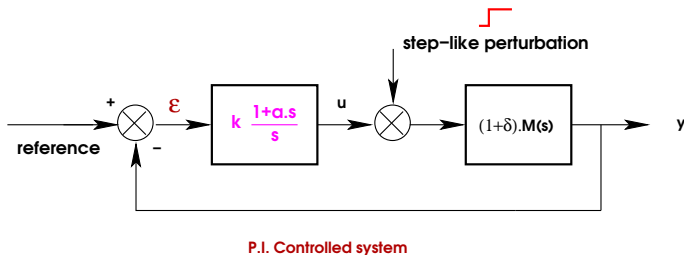
Outline

1. Basic definitions

2. A classical approach to robust control
   - Standard robustness margins
   - Limitations
   - The module margin

3. Towards a modern approach

4. Setting and solving an $\mathcal{H}_\infty$ problem

5. Conclusion

6. Application to a double-integrator system
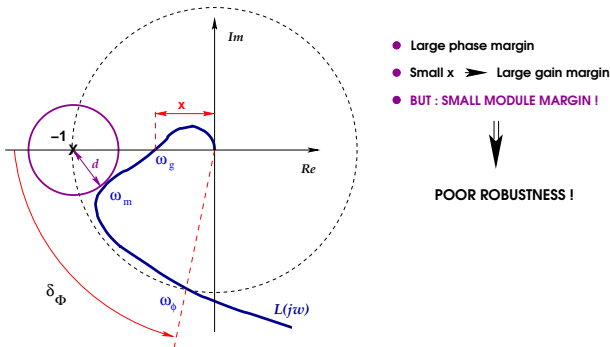
Standard robustness margins

Robustness has always been a central issue in Automatic Control. Because of computers limitations, very simple notions were initially introduced and are still used today:

- Gain margin : $L(\omega_g) + \Delta_g^* = (1 + \delta_g)L(\omega_g) = -1$
- Phase/delay margins : $L(\omega_\phi) + \Delta_\phi^* = L(\omega_\phi)e^{-j\delta_\phi} = -1$
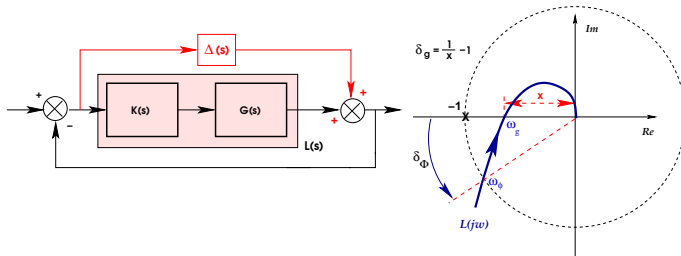
Limitations: Illustration in the Nyquist plane

Despite large phase and gain margins the Nyquist locus may get dangerously close to the critical point...

- Large phase margin
- Small x ➤ Large gain margin
- BUT : SMALL MODULE MARGIN !

⇓

POOR ROBUSTNESS !

$$\begin{cases} L(\omega_m) + \Delta_m^* = (1 + \delta_m)L(\omega_m)e^{-j\delta_{\phi_m}} = -1 \,, \ |\Delta_m^*| = d \\ \delta_m << \delta_g \,, \ \delta_{\phi_m} << \delta_\phi \end{cases}$$

The module margin

As illustrated by the previous slide, a new robustness margin should then be introduced and will be defined as the minimal distance d between the Nyquist locus and the critical point. This distance is refered to as the module margin.

Exercise

With the above notation, show that :

$$d = \frac{1}{\sup_w |S(j\omega)|} \quad with \quad S(j\omega) = \frac{1}{1 + L(j\omega)}$$

and find a physical interpretation for the transfer function $S(s)$...

Outline

1. Basic definitions

2. A classical approach to robust control

3. Towards a modern approach
   - Extension to MIMO systems
   - Singular values
   - The $\mathcal{H}_\infty$ norm
   - The small gain theorem
   - Time-domain interpretation
   - A robust prerformance problem

4. Setting and solving an $\mathcal{H}_\infty$ problem

Extension to MIMO systems

The "modern" approach - developed at the end of the 1980's - to robust control is essentially based on a multi-variable extension of the module margin.

As is clear from above, in the SISO case, the distance to singularity can be obtained equivalently by computing the module of a specific transfer function or directly by solving at each frequency the equation:

$$1 + G(j\omega)K(j\omega) + \Delta(\omega) = 0 , \ \ |\Delta(\omega)| = d(\omega)$$

In the MIMO case, the singularity of a matrix is evaluated through its determinant, and the problem "simply" becomes:

$$det(I + G(j\omega)K(j\omega) + \Delta(\omega)) = 0$$

Extension to MIMO systems

The above extension is conceptually simple. However it leads to numerically untractable problem. Basically, the idea of robust control, consists indeed to compute a controller $K(s)$ such that for a given uncertainty level γ which bounds a size (to be precised) of $\Delta(s)$ we have:

$$det(I + G(j\omega)K(j\omega) + \Delta(\omega)) \neq 0 \quad (*)$$

for all admissible uncertainties γ-bounded Δ.

Two problems appear there:

- How can we obtain a numerically tractable condition which guarantees (*),
- How can we simply measure the size of the uncertainty matrix?

The singular value notion will provide an elegant answer to both problems...

Singular values

Definitions

$$\sigma_i(M) = \sqrt{\lambda_i(M^*M)}\,, \quad \overline{\sigma}(M) = \max_{i=1\ldots n}\{\sigma_i\}\,, \quad \underline{\sigma}(M) = \min_{i=1\ldots n}\{\sigma_i\}$$

Properties

- $\overline{\sigma}(M_1 M_2) \leq \overline{\sigma}(M_1)\overline{\sigma}(M_2)$
- $\overline{\sigma}(M^{-1}) = \frac{1}{\underline{\sigma}(M)}$
- $\overline{\sigma}(M) < 1 \;\Rightarrow\; det(I + M) \neq 0$
- $\underline{\sigma}(M) > 1 \;\Rightarrow\; det(I + M) \neq 0$

Singular values can be viewed as a "matricial" generalization of the absolute value of a complex number.

The $\mathcal{H}_\infty$ norm

Definition

$$||G(s)||_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$$
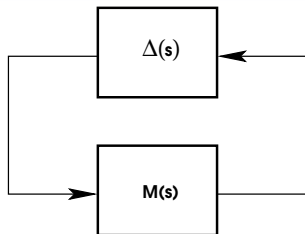
A first application

Show that the minimization of the $\mathcal{H}_\infty$ norm of the following MIMO system:

$$S(s) = (I + G(s)K(s))^{-1}$$

tends to "push" the matrix $I + G(j\omega)K(j\omega)$ "away" from singularity.

The small gain theorem

Consider the following interconnection:

where $M(s)$ and $\Delta(s)$ are stable linear systems of compatible dimensions. Then the interconnection is internally stable for all Δ such that $||\Delta(s)||_\infty \leq 1/\gamma$ **if and only if**:

$$||M(s)||_\infty < \gamma$$

Application of the small gain theorem

Give an expression of the transfer to be minimized so as to improve the robustness of the controller versus additive uncertainties on the plant.
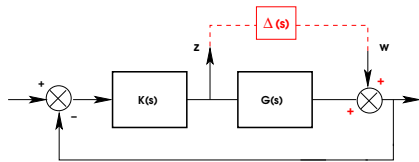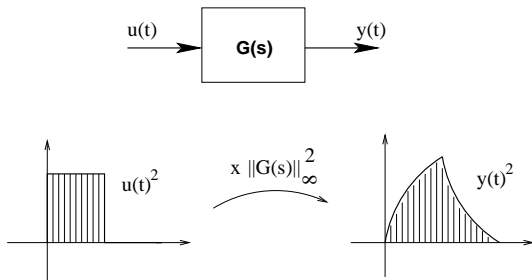

The transfer "seen" by Δ is given by:

$$T(s) = \mathcal{T}_{w \to z} = -(I + K(s)\,G(s))^{-1}K(s)$$
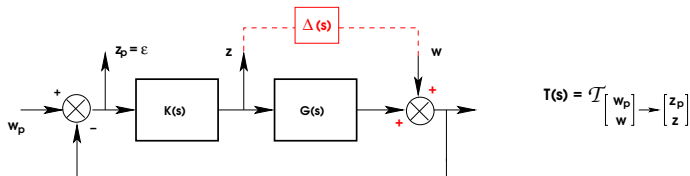
Time-domain interpretation

The $\mathcal{H}_\infty$ norm has an interesting time-domain interpretation which makes it useful to handle peformance problems as well.

$$\|G(s)\|_\infty^2 = \sup_{u \in \mathcal{L}_2} \frac{\int_0^\infty y(t)^2 \, dt}{\int_0^\infty u(t)^2 \, dt}$$

Specifying a robust performance problem

Consider the following figure:

which defines an augmented transfer function $T(s)$. Assume that $||T(s)||_\infty < \gamma$, then from the small-gain theorem and using the time-domain interpretation of the $\mathcal{H}_\infty$ norm, it is readily checked that :

$$\forall \Delta, ||\Delta(s)||_\infty < 1/\gamma \ \Rightarrow \ \int_0^\infty \epsilon^2(t)\,dt \ < \gamma^2 \int_0^\infty w_p^2(t)\,dt$$

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2. A classical approach to robust control

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4. Setting and solving an $\mathcal{H}_\infty$ problem
   - The standard form
   - Weighting functions
   - Resolution

5. Conclusion

6. Application to a double-integrator system

The standard form

The standard form $P(s)$ is an interconnected plant which describes in a compact and uniform setting the $\mathcal{H}_\infty$ problem to be solved:
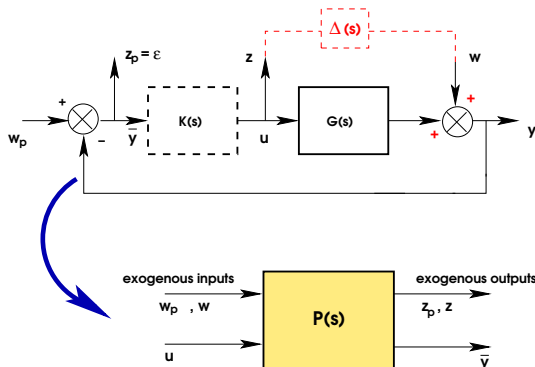
Weighting functions

It is usually impossible and not realistic anyway to minimize simultaneously the $\mathcal{H}_\infty$ norm of several transfer functions on the whole frequency range. Typically the performance level is only critical for low frequencies, while the robustness versus modelling errors should be guaranteed for higher frequencies. Such additional specifications are easily handled by weighting functions. For simplicity, first-order filters are often used.
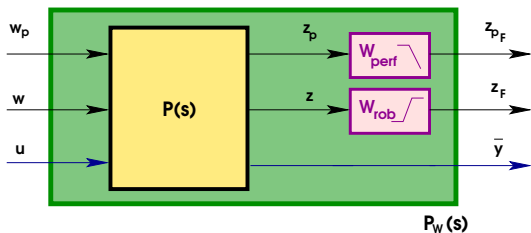
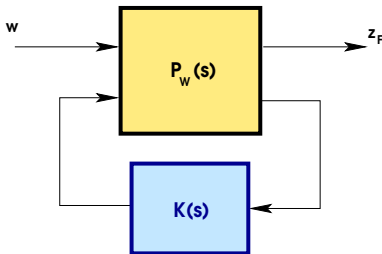**Problem formulation**

Given a standard plant $P(s)$ and weighting functions, the resolution of the $\mathcal{H}_\infty$ synthesis problem aims at finding a controller $\hat{K}(s)$ defined as follows :

$$\hat{K}(s) = arg \min_K \|\mathcal{F}_l(P_W(s), K(s))\|_\infty$$

where $\mathcal{F}_l(P_W(s), K(s))$ denotes the closed-loop transfer between exogenous inputs and outputs as illustrated below.

Standard resolution

- The above control problem was shown in the 1980's to be a **convex** problem provided that **the order of the controller is not constrained**: $K(s)$ and $P_W(s)$ have the same orders.

- From 1989, an efficient algorithm for solving $\mathcal{H}_\infty$ problems exist. It relies on the resolution of two coupled Riccati equations via a bissection algorithm implemented in Matlab.

- The application of this algorithm (**hinfsyn**) is restricted to "regular" standard plants which fulfill some technical assumptions. Advanced versions can be used which include a regularization routine.

LMI-based resolution

- In 1994, the $\mathcal{H}_\infty$ control problem was rewritten as a linear objective minimization problem over **LMI** constraints. Thanks to the flexibility of the LMI framework, regularity assumptions are no longer required. The associated Matlab routines are (**hinfric** and **hinflmi**). They are available with the LMI-LAB package which is now integrated to the Robust Control Toolbox.

- The LMI-based approach offers more flexibility, but is also numerically more demanding than the standard algorithm. As a result, it is still restricted today to low-order systems involving at most 20 to 30 states. The problem is that such a limit is rapidly reached with high-order weighting functions.

Resolution via non-smooth optimization

- In 2006, the $\mathcal{H}_\infty$ design problem was rewritten and solved as a nonsmooth optimization problem. This approach "works" directly with the parameters of the controller which can be **structured** as desired. The resulting problem is non convex.
- In recent versions of Matlab (2010b or higher), a new routine **hinfstruct** – based on non-smooth optimization - permits to compute **structured and fixed-order** $\mathcal{H}_\infty$ controllers. Although local solutions are obtained, this routine works very well in practice. As a well-known example, it now becomes possible to optimize PID gains so as to satisfy $\mathcal{H}_\infty$ constraints.
- More options will be available in the future which will probably give a second start to $\mathcal{H}_\infty$ control theory and will definitely make it popular in the industry.
- More information is available on Pierre Apkarian's homepage:
  <div style="text-align:center; color:red">http://pierre.apkarian.free.fr/</div>

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Conclusion

The $\mathcal{H}_\infty$ control technique provides an attractive approach to robust control design which permits not only to take robustness criteria into account, but performance requirements as well. However, despite powerful algorithms, such techniques still require a significant expertise of the control engineer who:

- has first to select the appropriate standard form according to its specific control problem
- who has then to set up and tune weighting functions correctly.

As a final point, it should be emphasized that the standard $\mathcal{H}_\infty$ control method which we have presented in this talk is not intended to be used to solve problems with highly structured uncertainties. For such applications, the method is too convervative and generally fails to give a relevant solution. For such cases, specific approaches should be considered (μ synthesis).

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