On convex characterizations of anti-windup controllers

with applications to the design of flight control systems

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Introduction

In this presentation, a new method is proposed to compute dynamic anti-windup compensators. The results are essentially based on:

- a recent description of dead-zone nonlinearities using generalized sector conditions,
- the minimization of a performance objective over a restricted class of input signals

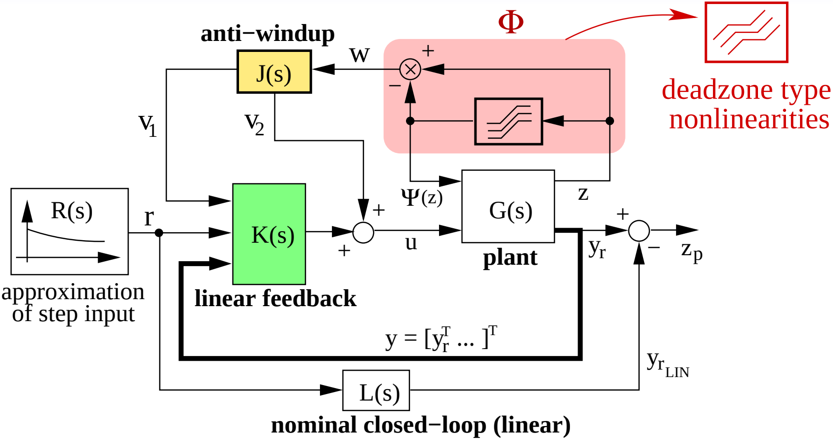
An original three-step procedure is proposed:

- Perform first a full-order anti-windup synthesis by convex optimization,
- Analyse the poles of the full-order compensator and select those which are located inside the band-with of the nominal closed-loop,
- Use the above selection to perform a reduced-order synthesis by fixing the poles.

Outline

1. Problem statement
   - The standard interconnection
   - Design objectives
   - Nonlinear closed-loop equations
2. Main theoretical results
3. Fighter aircraft application
4. Conclusion

Standard interconnection



Design objectives

Assuming that a linear feedback controller $K(s)$ was preliminarily designed, the issue is now to compute a dynamic anti-windup compensator $J(s)$:

$$\begin{cases} \dot{x}_J = A_J\, x_J + B_J\, w \in \mathbb{R}^{n_J} \\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C_J\, x_J + D_J\, w \in \mathbb{R}^{p_J} \end{cases} \tag{1}$$

such that:

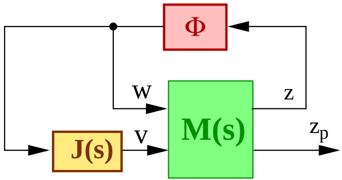
- the nonlinear closed-loop plant remains stable even for large reference inputs (r) This is ensured by maximizing the size of a stability domain in a given direction,
- the behaviour of the nonlinear system remains as close as possible to the nominal linear plant defined by $L(s)$. This is ensured by minimizing the energy of the error signal z_p.

The bounded reference inputs $r(t) \in \mathcal{W}_\tau^p(\rho)$ are generated with a "step-like" profile as follows:

$$R(s):\ \tau\dot{r} + r = 0\,,\ r(0) = r_0 \in \mathbb{R}^p\,,\ ||r_0|| \leq \rho \tag{2}$$

Nonlinear closed-loop equations (1/2)

The standard interconnection of slide 4 can be redrawn in a compact form as follows:



where, under mild assumptions, the augmented plant $M(s)$ has no feedthrough terms and then reads :

$$\left\{\begin{array}{l} \dot{\xi} = A\xi + B_\phi w + B_a v \ , \quad \xi = \begin{bmatrix} r \\ x_L \\ x_G \\ x_K \end{bmatrix} \in \mathbb{R}^{n_M} \\ z = C_\phi \xi \in \mathbb{R}^{m} \\ z_p = y_r - y_{r_{LIN}} = C_p \xi \in \mathbb{R}^{p} \end{array}\right. \tag{3}$$

Nonlinear closed-loop equations (2/2)

Then, the nonlinear closed-loop equations are finally obtained as :

$$\left\{ \begin{array}{l} \dot{x} = \begin{bmatrix} A & B_a C_J \\ 0 & A_J \end{bmatrix} x + \begin{bmatrix} B_\phi + B_a D_J \\ B_J \end{bmatrix} \phi(z) \\ z = \begin{bmatrix} C_\phi & 0 \end{bmatrix} x \\ z_p = \begin{bmatrix} C_p & 0 \end{bmatrix} x \end{array} \right. \tag{4}$$

where the global state vector x can be partitionned as:

$$x = \begin{bmatrix} \xi \\ x_J \end{bmatrix} = \begin{bmatrix} r \\ \zeta \end{bmatrix} \in \mathbb{R}^n \text{ with } \zeta = \begin{bmatrix} x_L \\ x_G \\ x_K \\ x_J \end{bmatrix} \in \mathbb{R}^{n-p}$$

Note that in our approach the reference input signal $r(t)$ is viewed as a part of the state-vector.

Outline

1 Problem statement

2 Main theoretical results
- Performance characterization
- Full-order synthesis
- Reduced-order synthesis

3 Fighter aircraft application

4 Conclusion

Performance chatacterization (1/2)

Consider the above nonlinear closed-loop plant with a given anti-windup controller $J(s)$. If there exist matrices $Q \in \mathbb{R}^{n\times n}$, $S = diag(s_1, \ldots, s_m)$, $Z \in \mathbb{R}^{m\times n}$ and positive scalars γ and $\overline{\rho}$ such that the following LMI conditions hold:

$$\begin{pmatrix} Q & \star \\ \begin{bmatrix} \overline{\rho}\, I_p & \mathbf{0} \end{bmatrix} & I_p \end{pmatrix} > 0 \qquad (5)$$

$$(\star) + \begin{pmatrix} \begin{bmatrix} A & B_a \mathbf{C_J} \\ 0 & \mathbf{A_J} \end{bmatrix} Q & \begin{bmatrix} B_\phi \\ \mathbf{B_J} \end{bmatrix} S & 0 \\ S\mathbf{D_J}^T \begin{bmatrix} B_a^T & 0 \end{bmatrix} - Z & -S & 0 \\ \begin{bmatrix} C_p & 0 \end{bmatrix} Q & 0 & -\frac{\gamma}{2} I_p \end{pmatrix} < 0 \qquad (6)$$

$$\begin{pmatrix} Q & \star \\ Z_i + \begin{bmatrix} C_{\phi_i} & 0 \end{bmatrix} Q & 1 \end{pmatrix} > 0\,, \quad i = 1 \ldots m \qquad (7)$$

Performance chataterization (2/2)

then, for all $\rho \leq \overline{\rho}$, and all reference signals $r(t) \in \mathcal{W}_\tau^p(\rho)$, the nonlinear interconnection is stable for all initial condition ζ_0 in the performance domain $\mathcal{E}(\rho)$ defined as follows :

$$\mathcal{E}(\rho) = \left\{ \zeta \in \mathbb{R}^{n-p} / \forall r \in \mathcal{W}_\tau^p(\rho), \begin{bmatrix} r \\ \zeta \end{bmatrix}^T P \begin{bmatrix} r \\ \zeta \end{bmatrix} \leq 1 \right\} \quad (8)$$

with $P = Q^{-1}$.

Moreover, the energy of the tracking error z_p satisfies :

$$\int_0^\infty z_p(t)^T z_p(t)\, dt \leq \gamma \quad (9)$$

Full-order anti-windup synthesis (1/3)

Let us now focus on the design issue. In such a case, the analysis variables (Q,S and Z) introduced in the above performance analysis result, and the compensator variables A_J, B_J, C_J and D_J have to be optimized simultaneously.

Consequently, the inequality (6) becomes a **BMI** and is therefore *a priori* no longer convex.

Nevertheless, following a standard approach, in the full-order case, the synthesis variables A_J, B_J,... are easily eliminated thanks to the projection lemma and convexity is then recovered.

This result is summarized in the following slide where N_a denotes any basis of the nullspace of B_a^T and $u(\overline{\rho}) = [\overline{\rho} I_p \ 0]^T \in \mathbb{R}^{n_M \times p}$.

Full-order anti-windup synthesis (2/3)

There exists a compensator $J(s)$ satisfying (5),(6),(7) **iff** $\exists X = X^T, Y = Y^T \in \mathbb{R}^{n_M \times n_M}$, $S = diag(s_1, \ldots, s_m)$, $U, V \in \mathbb{R}^{m \times n_M}$ such that the following LMI conditions hold:

$$u(\overline{\rho})^T X u(\overline{\rho}) < I_p \tag{10}$$

$$\begin{pmatrix} A^T X + XA & \star \\ C_p & -\gamma I_p \end{pmatrix} < 0 \tag{11}$$

$$\begin{pmatrix} N_a^T (AY + YA^T) N_a & \star & \star \\ N_a & -2S & \star \\ C_p Y N_a & 0 & -\gamma I_p \end{pmatrix} < 0 \tag{12}$$

$$\begin{pmatrix} X & \star & \star \\ I_{n_M} & Y & \star \\ U_i & V_i + C_{\phi_i} Y & 1 \end{pmatrix} > 0, \quad i = 1 \ldots m \tag{13}$$

Full-order anti-windup synthesis (3/3)

Since the compensator variables have disappeared in the above existence conditions (11),(12), they have to be computed as follows:

1. Compute Q from X and Y :

$$Q = \begin{pmatrix} Y & I \\ N & 0 \end{pmatrix} \begin{pmatrix} I & X \\ 0 & M \end{pmatrix}^{-1} \text{ with } M^T N = I_{n_M} - XY$$

2. Fix Q, S, Z and solve the convex feasibility problem (6) *w.r.t.* the design variables A_J, B_J, C_J and D_J :

$$(\star) + \begin{pmatrix} \begin{bmatrix} A & B_a \mathbf{C_J} \\ 0 & \mathbf{A_J} \end{bmatrix} Q & \begin{bmatrix} B_\phi \\ \mathbf{B_J} \end{bmatrix} S & 0 \\ S\mathbf{D_J}^T \begin{bmatrix} B_a^T & 0 \end{bmatrix} - Z & -S & 0 \\ \begin{bmatrix} C_p & 0 \end{bmatrix} Q & 0 & -\frac{\gamma}{2} I_p \end{pmatrix} < 0 \qquad \text{(6b)}$$

Reduced-order anti-windup synthesis

For practical reasons (due to implementation aspects), it is most often required to limit the complexity of the anti-windup compensator. Moreover, the full-order design approach does not permit to control the location of the poles, which is often useful as well.

Our proposed reduced-order synthesis approach will permit to control both the order of the compensator and the location of its poles by fixing the A_J matrix. Moreover, the convexity is then ensured by the following proposition:

Proposition : *The BMI constraint (6) is convex as soon as the design variables A_J and C_J are fixed.*

which immediately follows from a standard change of variables: $\tilde{B}_J = B_J S$, $\tilde{D}_J = D_J S$.

Reduced-order compensators' basis

Fixing the matrices A_J and C_J may appear as a difficult task for the designer. However, fixing the poles is much more intuitive and finally turns out to be equivalent. Let us decompose indeed $J(s)$ as follows:

$$J(s) = M_0 + \sum_{i=1}^{n_1} \frac{M_{i1}}{s+\lambda_i} + \sum_{i=1}^{n_2} \frac{M_{i2}}{s^2 + 2\eta_i\omega_i + \omega_i^2} \tag{14}$$

and fix the matrices A_J and C_J as indicated below:

$$\begin{aligned}
&\bullet\ A_J = \text{diag}\left(-\lambda_1, \ldots, -\lambda_{n_1}, A_1, \ldots, A_{n_2}\right) \\
&\bullet\ C_{J_k} = \Big[\underbrace{1 \ldots 1}_{n_1} \underbrace{[1\ 0] \ldots [1\ 0]}_{n_2}\Big],\ k = 1 \ldots p_J \\
&\text{with}:\ A_i = \begin{pmatrix} 0 & 1 \\ -\omega_i^2 & -2\eta_i\omega_i \end{pmatrix},\ i = 1 \ldots n_2
\end{aligned} \tag{15}$$

then, there exist $B_J = f(M_{ij})$ and $D_J = M_0$, such that:

$$J(s) = C_J(sI - A_J)^{-1}B_J + D_J$$

Reduced-order anti-windup algorithm

Anti-windup synthesis with fixed-dynamics

1. Perform a full-order synthesis:
   1. Fix ρ and minimize γ under the LMI constraints (10), (11), (12), (13).
   2. Compute the Q matrix and the compensator $J(s)$
2. Perform a reduced-order synthesis:
   1. Select a set of relevant poles from the full-order compensator,
   2. build the matrices A_J and C_J of the reduced-order compensator,
   3. minimize γ under the LMI constraints (5), (6), (7) *w.r.t* the variables Q, S, Z, $\tilde{B}_J$, $\tilde{D}_J$.
   4. Compute B_J and D_J by inverting the aforementioned change of variables.

Outline

1. Problem statement
2. Main theoretical results
3. Fighter aircraft application
   - Open-loop model
   - Closed-loop interconnection
   - Design specifications
4. Conclusion

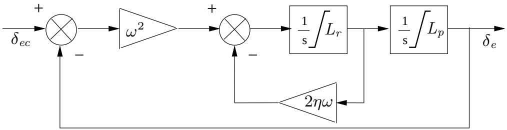
Open-loop model

The considered longitudinal fighter aircraft model is composed of two states (α is the angle-of-attack and q the pitch rate):

$$\begin{pmatrix} \dot{\alpha} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -0.5 & 1 \\ 0.8 & -0.4 \end{pmatrix} \begin{pmatrix} \alpha \\ q \end{pmatrix} + \begin{pmatrix} -0.2 \\ 5 \end{pmatrix} \delta_e$$

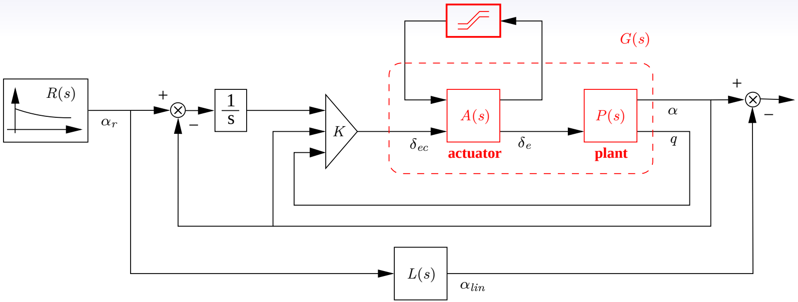
Note that a critical point in the flight envelope is chosen, for which the plant is open-loop unstable ($Mach = 0.3$, $H = 5000\, ft$).

A second-order actuator is then introduced ($\eta = 0.6$ and $\omega = 60\, rad/s$):



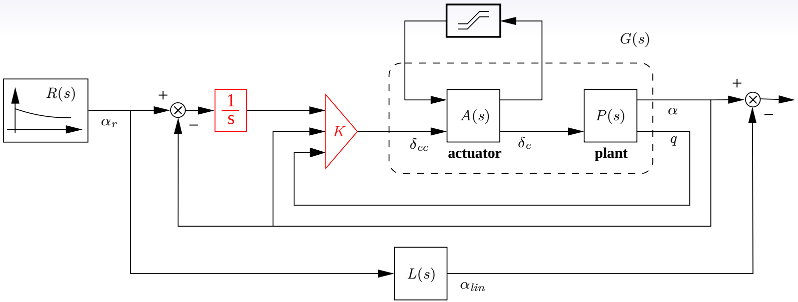
Magnitude and rate saturations appear in limited integrators ($L_p = 20\, deg$ and $L_r = 80\, deg/s$).

Closed-loop interconnection



- open-loop saturated plant
- nominal linear controller
- nominal closed-loop plant
- reference input

Closed-loop interconnection



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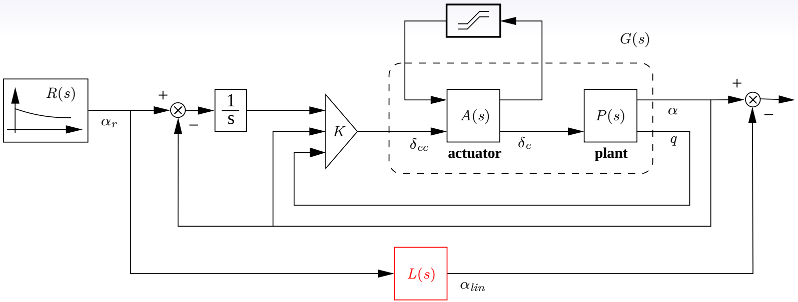
Closed-loop interconnection



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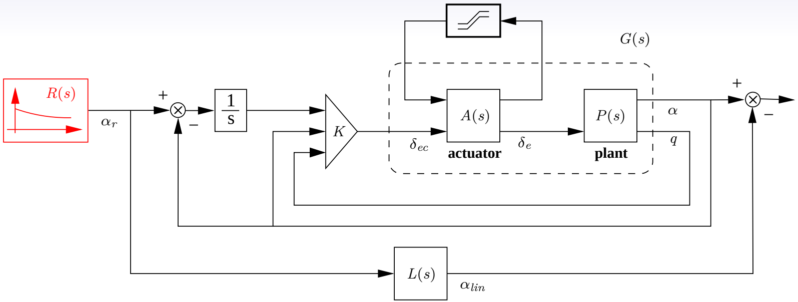
Closed-loop interconnection



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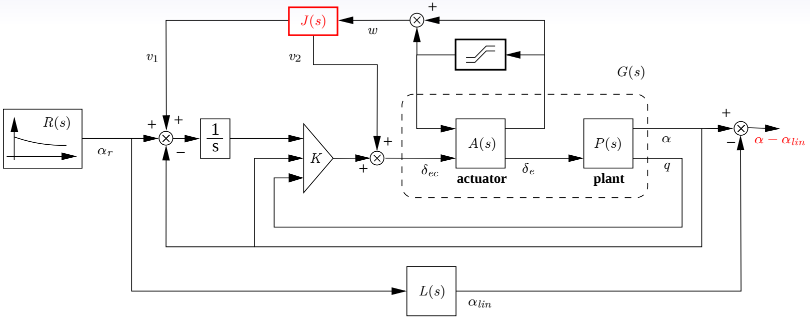
Design specifications

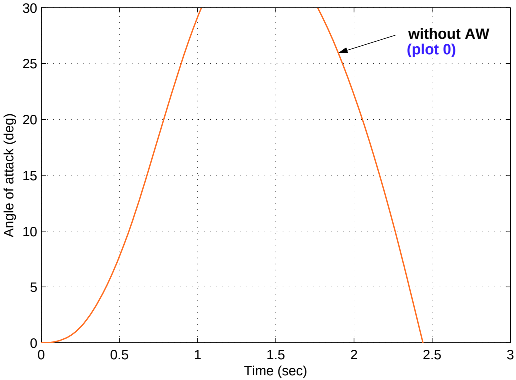


Design objective

Design a dynamic anti-windup controller $J(s)$ such that the energy of the tracking error $\alpha - \alpha_{lin}$ is minimized.

Numerical results

Time-domain responses to a 20 deg step command in angle of attack.



Plot 0 - orange line

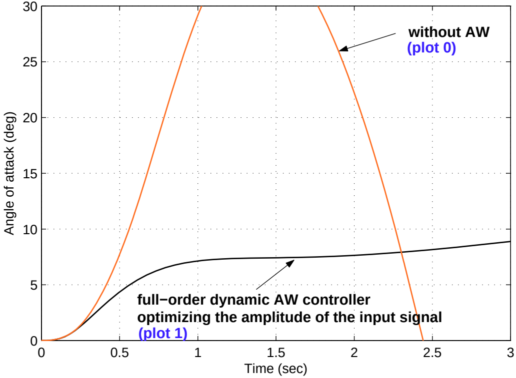
Without anti-windup

Unstable plant

The PID controller performs well as long as the amplitude of α_r does not exceed $7\ deg$. Beyond this value, a performance degradation appears and stability is finally lost when the amplitude gets larger than $7.8\ deg$.

Numerical results

Time-domain responses to a 20 deg step command in angle of attack.



Plot 1 - black line

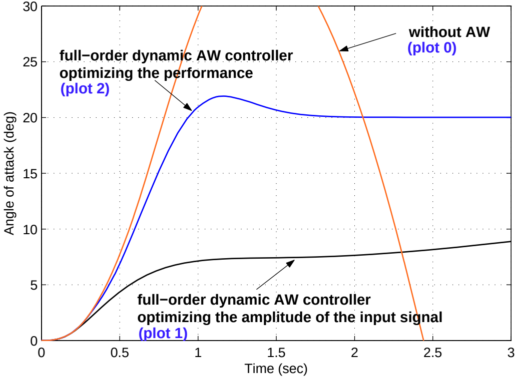
Full-order anti-windup (stability)

Stable plant, but very poor performance level

The maximum value of α_r is computed, for which the plant is guaranteed to be stable. A very large value is obtained $(29.6\ deg)$, which means that the stability domain is considerably enlarged.

Numerical results

Time-domain responses to a 20 deg step command in angle of attack.



Plot 2 - blue line

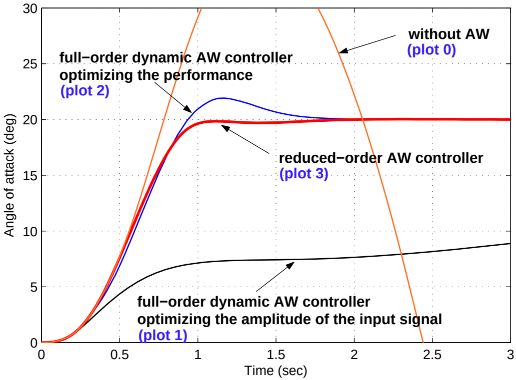
Full-order anti-windup (performance)

Good step response, but small overshoot

The energy γ of the error signal $\alpha - \alpha_{lin}$ is minimized, where α_{lin} is the step response of the unsaturated closed-loop plant. A reasonably small value is obtained (0.11)

Numerical results

Time-domain responses to a 20 deg step command in angle of attack.



Plot 3 - red line

Reduced-order anti-windup with fixed poles

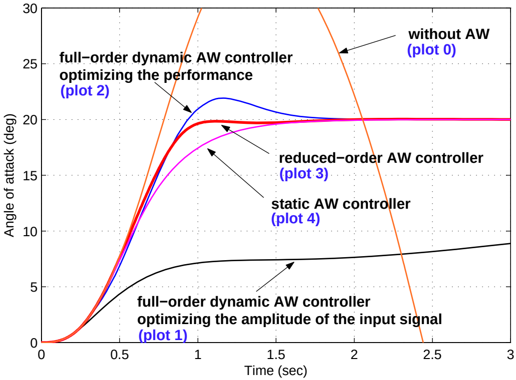
Fast response with no overshoot

The poles of the full-order anti-windup controller are analyzed. A selection is made and a reduced-order controller is computed.

$$-0.0013 \quad -0.45 \quad -1.80 \quad -4.21 \quad -5.53 \pm 3.23j \quad -548 \quad -4750$$

Numerical results

Time-domain responses to a 20 deg step command in angle of attack.



Plot 4 - purple line

Static anti-windup (performance)

Good time response, but a bit slow

Finally, for the purpose of comparison, a static anti-windup controller is synthesized. Such a design can be viewed as a special case of the previous one. The controller here reduces to the feedthrough term D_J.

Outline

Conclusion

In this talk, some new results have been presented to compute dynamic anti-windup controllers by convex optimization of an original performance objective over a restricted class of input signals. Both the full-order and reduced-order cases have been considered.

The applicability of the results has then been demonstrated on a realistic example which has also clearly highlighted the interest of :

- the performance optimization compared to the maximization of the amplitude of the input signal,
- fixing the poles of the anti-windup controller (by selecting a part of those which were placed by the full-order design) to avoid slow dynamics.

Future works will be devoted to the improvement of the robustness properties of the anti-windup compensators so as to facilitate their implementation.