Limit-cycles prevention via multiple $\mathcal{H}_\infty$ constraints with an application to anti-windup design

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INTRODUCTION (1/2)

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INTRODUCTION (1/2)

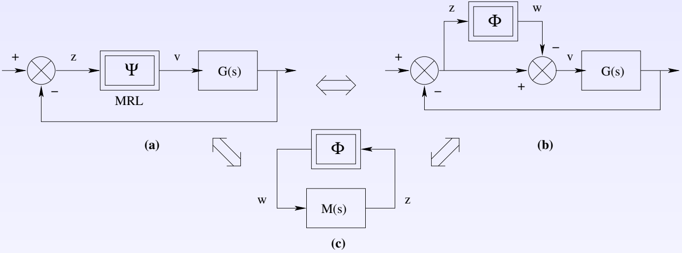
- Stability analysis of nonlinear control systems can often be treated as a problem of investigating the existence of sustained oscillations known as limit-cycles,
- Although the detection of such oscillatory behavior cannot be regarded as a rigorous stability test, the knowledge of their existence and characteristics (magnitude and frequency) is often crucial to predict wether the plant will become unstable or not,
- A variety of techniques have then been developed in the past decades to detect and avoid limit-cycles among which describing function (DF) methods are still very popular today thanks to their close connections with linear frequency-domain techniques. These techniques offer possibilities for systematic control systems design in the presence of input saturations which are often present in practice.

INTRODUCTION (1/2)

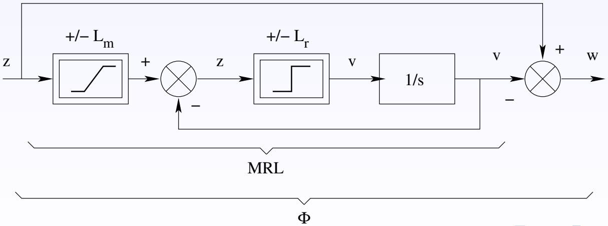
- Inspired by such results, the central contribution of this paper is based on these classical frequency-domain conditions to check the existence of limit-cycles in magnitude and rate limited control systems. The main result consists of a new characterization of the above conditions via multiple $\mathcal{H}_\infty$ constraints from which a new anti-windup design approach can be derived,
- The talk is organized as follows :
  - Describing functions and necessary condition of oscillation
  - Avoiding limit cycles
  - Application to anti-windup design
  - Illustration

DESCRIBING FUNCTIONS

Considered class of nonlinear systems :



where the nonlinear operators Ψ and Φ are described by :



DESCRIBING FUNCTIONS

In the context of sinusoidal-input describing function (SIDF) analysis, the input signal $z(t)$ is assumed to be $z(t) = x \sin \omega t$. The output $v(t) = \Psi(x \sin \omega t)$ is then a periodic signal and the describing function gain is defined as the fundamental of the Fourier series of $v(t)$ divided by the input magnitude x. Denoted $N_\Psi^{(k)}$, the resulting complex-valued gain reads (for the k^{th} harmonic) :

$$N_\Psi^{(k)}(x,\omega) = \frac{j\omega}{\pi x} \int_0^{\frac{2\pi}{\omega}} \Psi(x \sin \omega t) e^{-jk\omega t} dt \qquad (1)$$

and the existence of limit-cycle oscillations ($x_c \sin \omega_c t$) in the nonlinear closed-loop plant can be investigated through the resolution of the harmonic balance equation :

$$1 + G(j\omega_c) N_\Psi^{(1)}(x_c,\omega_c) = 0 \Leftrightarrow M(j\omega_c) = 1/N_\Phi^{(1)}(x_c,\omega_c) \qquad (2)$$

The resolution of the above equation is usually performed graphically in the Nyquist plane by looking for intersections between the Nyquist plot of $M(s)$ and the critical loci of the nonlinearity $1/N_{\Phi}^{(1)}(x,\omega)$.

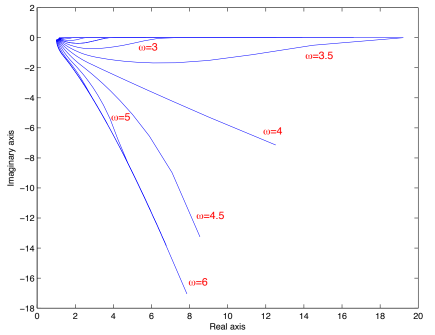


Figure : Critical loci $1/N_{\Phi}^{(1)}(x,\omega)$ in the Nyquist plane ($L_r = 1$, $L_m = 0.25$).

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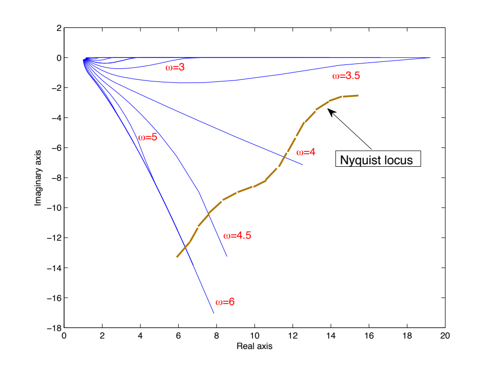


Figure : Critical loci $1/N_\Phi^{(1)}(x,\omega)$ and portion of Nyquist plot for $M(s)$

To avoid limit-cycles, no intersection should appear :

$$\forall \omega \geq 0, \forall x \geq 0, \quad M(j\omega) \neq 1/N_\Phi^{(1)}(x,\omega) \tag{3}$$

Since $|1/N_\Phi^{(1)}(x,\omega)| > 1$, $\forall \omega \geq 0, \forall x \geq 0$, a standard but conservative constraint to enforce (3) is :

$$\forall \omega \geq 0, |M(j\omega)| < 1 \Leftrightarrow \|M(s)\|_\infty < 1 \tag{4}$$

It can be relaxed as follows :

$$\begin{cases} \|M(s)\|_\infty < c_1 \\ \forall \omega \geq 0, |M(j\omega) - \alpha| > \rho \end{cases} \quad \text{with} \quad \begin{cases} c_1 > 1 \\ \alpha > c_1 \\ \rho \geq \alpha - 1 \end{cases} \tag{5}$$

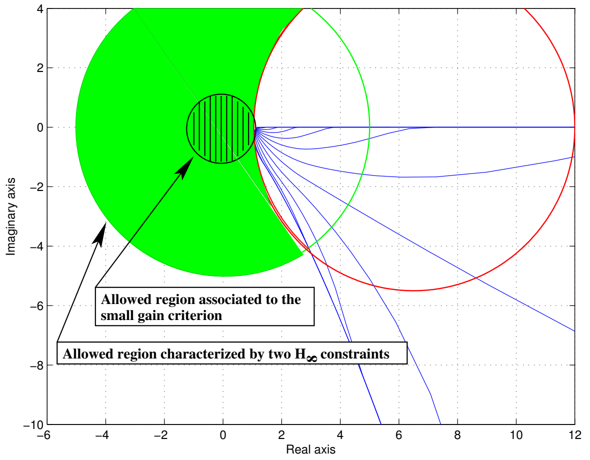


Figure : Comparison of constraints (4) and (5)

AVOIDING LIMIT-CYCLES

With the following lemma :

Lemma

Given $\alpha > 0$ and a stable LTI model $M(s)$, such that $\|M(s)\|_\infty < \alpha$, then the linear feedback interconnection $T(s) = (I - \alpha^{-1}M(s))^{-1}$ is well-posed and for any positive real c, the $\mathcal{H}_\infty$ constraint :

$$\|T(s)\|_\infty = \|(I - \alpha^{-1}M(s))^{-1}\|_\infty < c \qquad (6)$$

guarantees that the Nyquist plot of $M(s)$ remains **outside** the disk $\mathcal{D}(\alpha, \alpha/c)$ with center α and radius $\rho = \alpha/c$.

conditions (5) take the form of a **multi-channel $\mathcal{H}_\infty$ problem** :

$$\begin{cases} \|M(s)\|_\infty < c_1 \\ \|T(s)\|_\infty < c_2 \end{cases} \quad \text{with} \quad \begin{cases} 1 < c_1 < \alpha \\ c_2 \leq \frac{\alpha}{\alpha - 1} \end{cases} \qquad (7)$$

Remark

It is not always possible in practice to avoid limit-cycles, especially for **open-loop unstable plants**. In this case, the critical loci and the nyquist plot will generally intersect near 1 which cannot be excluded by the "red circle". In other words, c_2 cannot be minimized to satisfy the constraint $c_2 \leq \frac{\alpha}{\alpha-1}$.

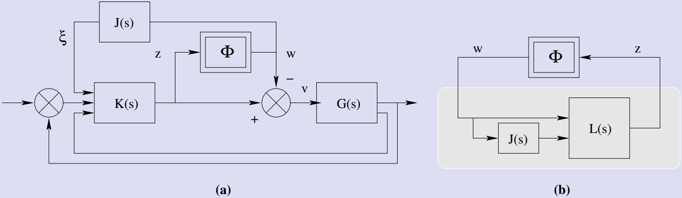
However, by exploiting more precisely the geometry of the critical loci, a region $\mathcal{R}$ can be defined such that :

$$\forall \omega \geq 0, M(j\omega) \notin \mathcal{R} \Rightarrow x_c > x_L \text{ or } \omega_c > \omega_L \quad (8)$$

Interestingly, the exclusion constraint $M(j\omega) \notin \mathcal{R}$ can also be enforced by the $\mathcal{H}_\infty$ conditions (7) for an appropriate choice of c_1, c_2 and α.

APPLICATION TO ANTI-WINDUP DESIGN

The anti-windup general structure



$$M(s) = \mathcal{T}_{w \to z}(s) = L_1(s) + L_2(s)J(s) \tag{9}$$

Resolution via mutli-objective $\mathcal{H}_\infty$ design

$$\hat{J}(s) = arg \min_{J(s)} \max \left(\mu \| T_1(s) \|_\infty, \| T_2(s) \|_\infty\right) \tag{10}$$

$$\left\{ \begin{array}{l} T_1(s) = (L_1(s) + L_2(s)J(s)) \\ T_2(s) = \left(I - \alpha^{-1}(L_1(s) + L_2(s)J(s))\right)^{-1} \end{array} \right. \tag{11}$$

ILLUSTRATION

Consider the following second-order open loop model :

$$G(s) = \frac{1}{s(s+0.1)} \tag{12}$$

Next define a stabilizing PID controller for this plant :

$$u(t) = \int_0^t (\theta_c(\tau) - \theta(\tau))d\tau - 2.5\theta(t) - 2.4\dot{\theta}(t) \tag{13}$$

The nominal closed-loop (without saturations) reads :

$$\mathcal{T}_{\theta_c \to \theta}(s) = \frac{1}{(s+1)(s^2+1.5s+1)} \tag{14}$$

ILLUSTRATION : LIMIT-CYCLE

Introduce magnitude ($L_m = 0.25$) and rate ($L_r = 1$) limitations on the control signals. Limit-cycle oscillations appear for $\theta_c = 2.114$:

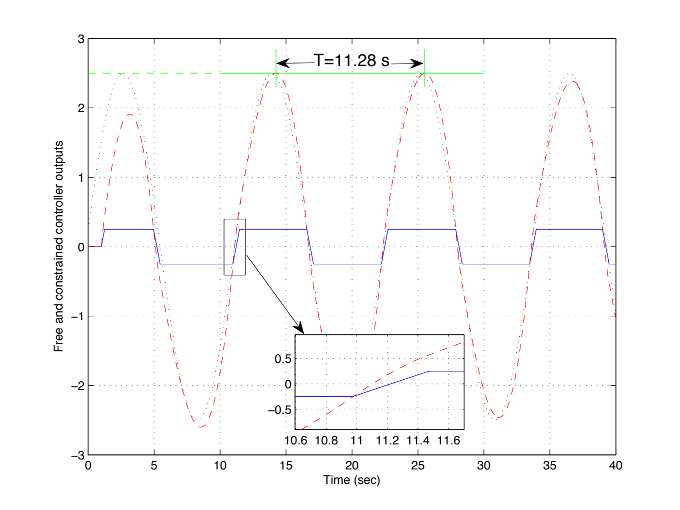
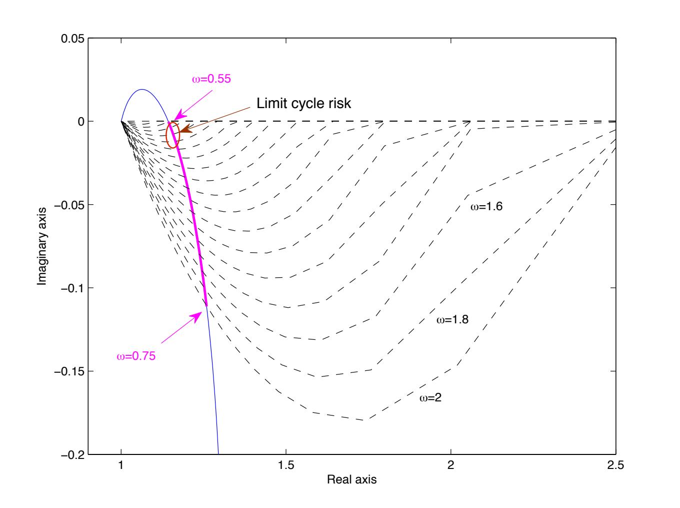


ILLUSTRATION : LIMIT-CYCLE

Limit-cycle is confirmed by Nyquist-plane analysis :



The PID control law is modified as follows :

$$u(t) = \int_0^t (\theta_c(\tau) - \theta(\tau) + \xi(\tau))d\tau - 2.5\theta(t) - 2.4\dot{\theta}(t) \qquad (15)$$

From this structure, the linear interconnection $L(s)$ is obtained as :

$$L(s) = [L_1(s) \;\; L_2(s)] = \frac{[1 + 2.5s + 2.4s^2 \quad - s(s + 0.1)]}{(1 + s)(1 + 1.5s + s^2)} \qquad (16)$$

and the multi-objective design problem is solved via nonsmooth optimization for $\alpha = 1.5$ and $\mu \geq 1$. This parameter is increased until $\mu \| T_1(s) \|_\infty = \| T_2(s) \|_\infty = 3.6$. Then, one obtains $\hat{J} = 1.53$.

ILLUSTRATION : RESULTS

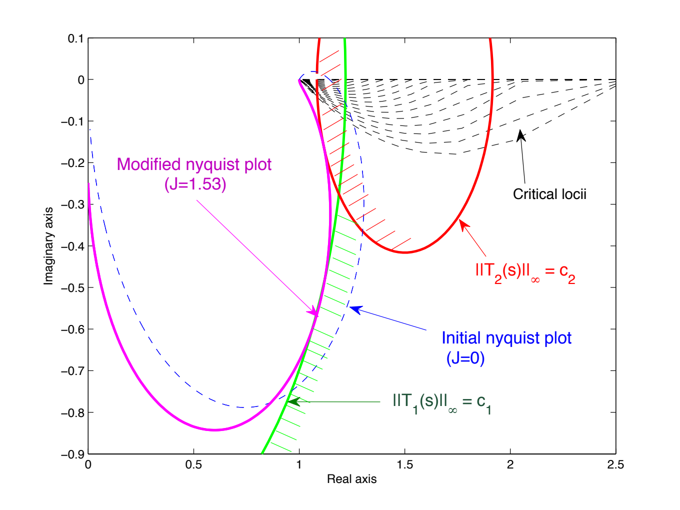
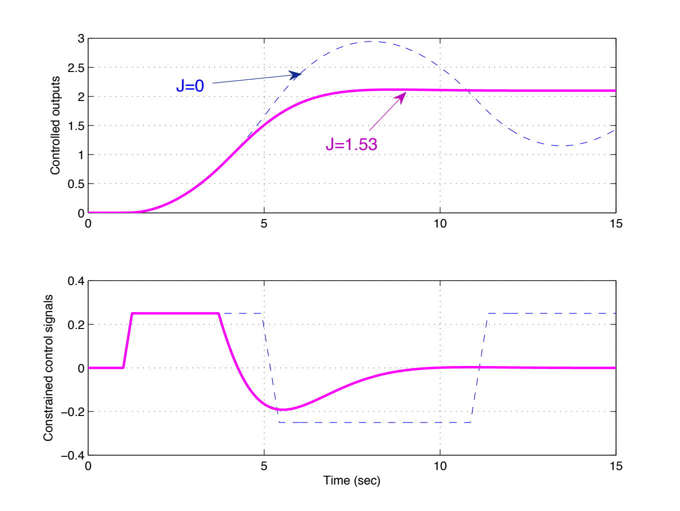


ILLUSTRATION : RESULTS



CONCLUSIONS AND FUTURE WORK

- Based on the well-known describing function approach, we have shown that limit cycles can be avoided as soon as H_∞ constraints are simultaneously satisfied by appropriately chosen linear interconnections,
- With this result, a new anti-windup design strategy can be derived for possibly high-order systems,
- The proposed framework is flexible and can be easily enriched to take into account additional $\mathcal{H}_\infty$ constraints.

Future work will be devoted to extensions of the anti-windup design approach to systems involving multiple and possibly more general nonlinearities. A specific attention will also be devoted to the tuning aspects so that the design procedure can be automatized and made available to non expert users.