INTRODUCTION TO THE
SKEW-µ TOOLBOX
WITH AERONAUTICAL APPLICATIONS

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Abstract:
In this note, a Toolbox which provides µ and skew µ methods for robustness analysis is presented. A special attention is focused on flexible systems for which a reliable computation of the robustness margin is a challenging problem. An important part of the paper is devoted to aeronautical illustrations. In a first example, a rigid missile with an $\mathcal{H}_\infty$-designed controller is to be analysed along its longitudinal axis. On this example, different algorithms are compared and a skew-µ technique is used to evaluate a robust performance level. The second example is devoted to a robust stability analysis problem on a flexible transport aircraft, for which specific techniques have been developed to handle high order systems with flexible modes.

Keywords: µ analysis - flexible systems - skew µ problems

1 Introduction

µ-analysis is now well-recognized as a classical and efficient robustness analysis technique. It has proved to be useful in many applications, and especially aeronautical ones. See e.g. [1], [11], [8], [10], see also [5] and included references. The issue is to validate the control law in the presence of unavoidable
model uncertainties, typically parametric uncertainties in the rigid aerodynamic model or in the flexible structural one, and also neglected dynamics (e.g. high frequency bending modes). Yet the application of the $\mu$ analysis technique remains difficult in some specific cases, which often occur in aeronautical sophisticated examples, such as:

- analysis of systems with real parametric uncertainties,
- analysis of high order systems with numerous flexible modes,
- robust performance analysis problems.

In the first class of problems, the main difficulty comes from the real nature of the uncertainties which often causes discontinuities in the $\mu$ plot. Then, many algorithms meet severe convergence difficulties. Fortunately, in most applications, real parametric and complex uncertainties are to be considered simultaneously. The latter are generally associated to neglected (high-frequency) dynamics or to performance blocks in skew-$\mu$ problems.

In the second class of problems, involving flexible systems, the $\mu$ plots tend to exhibit numerous high and narrow peaks, which might be difficult to detect by a classical gridding-based approach. In this case indeed, there exists a potential danger to miss the highest peak of the $\mu$ plot which would result in an over-evaluation of the robustness margin. Furthermore, flexible systems are also high-order systems, which might be a cause of numerical problems. For example, state-space based techniques will have to be avoided because of the computational burden.

Finally, the third class of problems is very common in practice but unfortunately cannot be solved properly by a standard $\mu$ analysis approach. As will be clarified further in the paper, a robust performance analysis problem can be either treated as:

- stability analysis problem in a specific region of the complex plane ($\Omega$-stability i.e modal performance),
- skew $\mu$ problem, for which specific computational tools are required ($H_\infty$ performance).

In our Skew-$\mu$ toolbox [6] (freeware for use with MATLAB®), a complete set of tools with user-friendly interfaces is proposed to tackle the above problems.
This freeware, which can be seen as a complement to [5] represents our own effort to bring some \( \mu \) analysis techniques from theory into reality. This especially concerns different works in [3, 2] and techniques described in [9, 7, 8, 10] (the list is not exhaustive).

The paper is organized as follows. In section 2, some technical preliminaries are first recalled. These include standard definitions and a discussion on computational difficulties. Then, in section 3, the content of the toolbox is presented. Finally, section 4 and 5 are devoted to realistic aeronautical applications, namely robustness analysis problems for a rigid missile and a flexible aircraft.

2 Some technical preliminaries

This section introduces the \( \mu \) framework. See [5] for a more detailed presentation.

2.1 The standard interconnection structure \( M - \Delta \)

![Figure 1: Standard interconnection structure (a) and LFT (b)](image)

The starting point of this toolbox is the standard interconnection structure \( M(s) - \Delta(s) \) of figure 1-a, (the obtention of such a structure is detailed in subsection 5.1.3. See also [5] for further details). Consider indeed an LTI closed loop subject to parametric uncertainties and neglected dynamics. It is most generally possible to transform a specific uncertain closed loop into this standard interconnection structure: the transfer matrix \( M(s) \) contains
the dynamics of the nominal closed loop (i.e. the closed loop without any model uncertainty) and the way the various model perturbations enter the closed loop. On the other hand, all model perturbations are gathered in the uncertain transfer matrix $\Delta(s)$, which has the following block diagonal structure:

$$\Delta(s) = \text{diag}(\Delta_1(s), \ldots, \Delta_m(s), \delta_1I_{q_1}, \ldots, \delta_nI_{q_n})$$

(1)

$\Delta(s)$ is called a structured model perturbation. $\Delta_i(s)$ is a block of neglected dynamics which is assumed to satisfy the $H_\infty$ constraint:

$$\|\Delta_i(s)\|_\infty \leq 1$$

(2)

$\delta_i$ is a real parametric uncertainty which is assumed to satisfy $\delta_i \in [-1, 1]$. These model uncertainties are obviously normalized. A template $W(s)$ is specified, so that the true neglected dynamics is $W(s)\Delta(s)$, while the uncertain parameter is $p = p^0 + \alpha\delta$. $p^0$ is the nominal value, and $\alpha$ is a weighting factor. If e.g. $\alpha = 0.1$, $\delta \in [-1, 1]$ means that $p$ varies between $\pm 10\%$.

Since the poles of $M(s)$ coincide with the poles of the nominal closed loop $M(s)$ is assumed to be asymptotically stable (no pole on the RHP or even on the imaginary axis). Let $B\Delta(s)$ the unit ball:

$$B\Delta(s) = \{\Delta(s) \ | \ \|\Delta(s)\|_\infty < 1\}$$

(3)

This means that all blocks of neglected dynamics satisfy (2) and all $\delta_i \in [-1, 1]$.

### 2.2 Robust stability analysis

With the above notations in mind, the robustness margin $k_m$ is defined as the maximal amount of model uncertainties, for which the closed loop is stable, i.e. the maximal value of $k$ for which the closed loop of figure 1 (and thus the original uncertain closed loop which was put under such a standard form) is stable for all $\Delta(s) \in kB\Delta(s)$. In the $\mu$ context the robustness margin $k_m$ is computed as:

$$k_m = \frac{1}{\max_{\omega \in [0, +\infty)} \mu(M(j\omega))} = \min_{\omega} \frac{1}{\mu(M(j\omega))}$$

(4)

$\mu(M(j\omega))$ is the s.s.v. associated to complex matrix $M(j\omega)$ and to structure (1) of the model perturbation. It is classically defined as:
\[
\mu_\Delta(M(j\omega)) = \frac{1}{\min(k / \exists \Delta \in kB\Delta \text{ with } \det(I - M(j\omega)\Delta) = 0)} \quad (5)
\]

= 0 if no \((k, \Delta)\) exists

Note that a block of neglected dynamics \(\Delta_i(s)\) becomes a full complex block at frequency \(\omega\), and that constraint (2) becomes at this frequency:

\[
\sigma(\Delta_i(j\omega)) \leq 1 \quad (6)
\]

Remarks:

(i) The notation \(\mu_\Delta(M(j\omega))\) indicates that this value simultaneously depends on complex matrix \(M(j\omega)\) and on the structure of the model perturbation \(\Delta\). For the sake of simplicity, we will often drop out the \(\Delta\) dependence, i.e. simply note \(\mu(M(j\omega))\).

(ii) The singularity of matrix \(I - M(j\omega)\Delta\) indicates the presence on the imaginary axis at \(j\omega\) of a pole of the standard interconnection structure \(M(s) - \Delta\). In this context, with reference to equation (4), the robustness margin, initially defined as the maximal amount of model uncertainties, for which the closed loop poles remain inside the LHP, can be reinterpreted as the size of the smallest destabilizing model perturbation, i.e. the one which brings one closed loop pole on the imaginary axis, i.e. on the border of the LHP.

2.3 Robust performance analysis

Performance can be defined in two different ways. In the case of a real model perturbation, a first solution is to study the robustness of the location of the closed loop poles despite parametric uncertainties. In the general context of a mixed model perturbation, a second and more classical solution consists in checking whether a frequency domain template on a closed loop transfer matrix remains satisfied despite model uncertainties. In the first issue, performance is rather defined in the time domain, whereas performance is defined in the frequency domain in the second one.

2.3.1 \(\Omega\) stability

In the case of a real model perturbation, it is possible to study the robustness of the location of the closed loop poles in other regions than the LHP. This
is especially the case of a truncated sector (see figure 2). Performance can
be defined in this context by minimal values $\xi_{\min}$ and $\alpha_{\min}$ for the damping
ratio $\xi$ and the degree of stability $\alpha$.

![image]

Figure 2: Robustness of a pole placement

In this context the robustness margin, defined as the maximal amount of
model uncertainties for which the closed loop poles remain inside the trunc-
cated sector, can still be computed with a frequency domain approach (4).
But the s.s.v. is computed on the border of the truncated sector, instead of
the border of the LHP, i.e. the imaginary axis.

Remarks:
(i) Nominal closed loop poles (i.e. those of $M(s)$) must belong to the trun-
cated sector.
(ii) Neglected dynamics are undefined outside the imaginary axis.

2.4 $H_\infty$ performance

In the spirit of $H_\infty$ control, performance is achieved if a closed loop transfer
matrix $T(s)$ satisfies a frequency domain template $W(s)$ at all frequencies $\omega$:

$$\sigma(W(j\omega)T(j\omega)) < 1 \quad (7)$$

Assume now the presence of uncertainties in the closed loop, so that $T(s)$ is
an LFT (i.e. the transfer between $w$ and $z$ in figure 1-b):

$$T(s) = F_l(M(s), \Delta_U(s))$$
\( \Delta_U(s) \) is most generally a mixed model perturbation, containing parametric uncertainties and neglected dynamics. Assume moreover that \( W(s) \) is included in \( F_i(M(s), \Delta_U(s)) \) to alleviate the notations. The nominal closed loop is assumed to satisfy the performance property at \( \omega \), i.e.:

\[
\sigma(F_i(M(j\omega), 0)) < 1
\]  

(8)

The robust performance problem consists in computing the maximal amount of uncertainties, for which closed loop performance is still achieved. The problem is thus to compute the maximal size of the mixed model perturbation \( \Delta_U(j\omega) \), for which the following relation remains satisfied:

\[
\sigma(F_i(M(j\omega), \Delta_U(j\omega))) < 1
\]  

(9)

Here again the problem is solved at each frequency \( \omega \). This robust performance problem can be equivalently transformed into an augmented robust stability problem, involving an additional fictitious full complex block (which is called a fictitious performance block) \( \Delta_P \). The standard interconnection structure \( M(s) - \Delta \) is then obtained with the augmented perturbation \( \Delta = \text{diag}(\Delta_P, \Delta_U) \).

Let us now define the skew s.s.v. (denoted \( \nu \)) associated to \( \Delta \), as follows:

\[
\nu(M(j\omega)) = \frac{1}{\min(k / \exists \Delta = \text{diag}(\Delta_P, k\Delta_U) \text{ with } \Delta \in B\Delta}
\]

and \( \text{det}(I - M(j\omega)\Delta) = 0 \)

(10)

With this definition in mind, it is easily seen that computing the maximal amount of uncertainties, for which closed-loop performance is achieved boils down to a skew \( \mu \) problem in which the fictitious performance block \( \Delta_P \) is maintained inside its unit ball while the size of the model perturbation \( \Delta_U \) is maximised.

2.5 Computational difficulties

• A common practice for solving (4) is to compute the s.s.v. \( \mu(M(j\omega)) \) at each point of a frequency gridding \( (\omega_i)_{i \in [1,N]} \). When choosing a sufficiently fine
frequency gridding, good results are obtained in many practical examples. Nevertheless a specific problem appears in the context of flexible systems: narrow and high peaks may indeed be obtained on the plot of $\mu(M(j\omega))$ as a function of frequency $\omega$. The use of a frequency gridding reveals unreliable in such a case: the risk is to miss a narrow and high peak on the $\mu$ plot, and thus to overevaluate the robustness properties of the closed loop (by underevaluating the value of the maximal s.s.v. over the frequency range). In the context of this new and difficult problem, we propose two methods are proposed in the Toolbox for computing a reliable estimate of $\mu(M(j\omega))$ as a function of $\omega$.

- Computing the exact value of the s.s.v. is an NP hard problem, so that the computational burden of the algorithms, which compute the exact value of $\mu$, is necessarily an exponential function of the size of the problem. It is consequently impossible in practice to compute the exact value of $\mu$ for large dimension problems. A usual solution is to compute $\mu$ upper and lower bounds instead of the exact value. The associated algorithms can be exponential-time (like the algorithms which compute the exact value of $\mu$), or more interestingly polynomial-time. Even if it is not possible to guarantee a priori the gap between the $\mu$ bounds when using polynomial-time algorithms, good results are usually obtained in practical realistic examples: this will be illustrated in the following.

- The practical usefulness of the $\mu$ bounds is now explained. For the sake of clarity, we restrict our attention to the case of a real model perturbation $\Delta = \text{diag}(\delta_i I_n)$. Let $D$ the unit hypercube:

$$D = \{ \delta = [\delta_1 \ldots \delta_n] \mid \delta_i \in \mathbb{R} \text{ and } |\delta_i| \leq 1 \}$$

(11)

$D$ corresponds to the unit ball in the specific context of a real model perturbation. An upper bound $\overline{\mu}$ of $\mu(M)$ gives a sufficient condition of nonsingularity of the matrix $I - M\Delta$, which is thus guaranteed to be nonsingular for all parametric uncertainties $\Delta$ inside $(1/\overline{\mu})D$. An upper bound $\overline{\mu}$ of the s.s.v. thus gives a lower bound $k_L$ of the robustness margin:

$$k_L = \min_{\omega \in [0, \infty)} \frac{1}{\overline{\mu}(M(j\omega))}$$

(12)

In the context of a robust stability problem in the presence of parametric uncertainties, robust stability of the closed loop can thus be guaranteed inside
the hypercube \( k_L D \) in the space of uncertain parameters.

Conversely, a lower bound \( \mu \) of \( \mu(M) \) gives a sufficient condition of singularity of the matrix \( I - M\Delta \), i.e. there exists a real model perturbation \( \Delta^* \in (1/\mu)D \), with \( I - M\Delta^* \) singular (in the context of a robust stability problem, \( \Delta^* \) is a destabilizing model perturbation). The usefulness of a \( \mu \) lower bound is twofold. As a first point, \( \mu \) gives a measure of the conservativeness of the upper bound \( \bar{\mu} \), by examining the tightness of the interval \([\mu, \bar{\mu}]\) which contains the exact value of \( \mu \). As a second point, an associated worst-case model perturbation \( \Delta^* \) is usually provided with \( \mu \) by the computational algorithm.

3 Content of the Toolbox

3.1 Architecture overview

The Toolbox is organized as shown by figure 3. It contains 6 directories. The main routines, to be further detailed in this section are listed on the figure.
Note also that the Toolbox includes several demo files related to different applications (missile, airplane, telescope mock-up). For further details on the use and description of the Toolbox, the reader is referred to [6].

3.2 Main tools

The main routines of the toolbox can be divided into three categories.

3.2.1 (skew) \( \mu \) upper-bound

Seven routines are devoted to \( \mu \) (or skew-\( \mu \)) upper-bound computation. A short description of the main routines is given below. Note that the last three routines are higher-level and can provide an upper-bound of \( \mu \) on a frequency range, while the first ones will only compute an upper-bound of \( \mu \) for a fixed frequency.

- **mu_zd.** This routine implements an exponential-time approach to compute an upper-bound of \( \mu \) in the case on non-repeated real uncertainties [3].

- **mu_ub.** Here a mixed-\( \mu \) upper-bound is computed by an LMI approach. The \( D-G \) scalings are associated to a single frequency point. However, their validity can be enforced on several points simultaneously. The algorithm is polynomial-time [4].

- **skew_mu_ub.** This routine is similar to the previous one but extended to skew uncertainties [9].

- **mu_frequency_gridding.** This is a high level routine to compute the robustness margin. According to the options, it may call one of the above routines (or routines from the \( \mu \) analysis and synthesis toolbox if available) at each point of a frequency gridding. Many types of robustness problems can be treated such as classical stability but also \( \Omega \)-stability and \( \mathcal{H}_\infty \) performance. The main drawback of this approach is that the margin is not guaranteed especially in the case of highly flexible systems.

- **mu_max_1.** This routine implements a recent approach [10, 7] to provide a guaranteed robustness margin on a frequency range. It is based on **mu_ub** or **skew_mu_ub** and a \( D-G \) scalings validation technique.
It is quite well suited to problems for which the $\mu$ plot exhibit numerous high and narrow peaks. Note that this routine can also be used for robust performance analysis problems. Yet, $\Omega$-stability analysis cannot be treated by this approach.

- **mu_max_2.** This routine also provides a guaranteed robustness margin. The technique [9] is however significantly different from the previous one. Here, on a given range, the frequency is extracted and treated as repeated real uncertainty. Therefore, this method cannot be applied to high-order systems because of the computational burden.

3.2.2 (skew) $\mu$ lower-bound

- **mu_dailey.** This routine implements an exponential-time technique to compute a $\mu$ lower-bound at a given frequency point, in the case of non-repeated real uncertainties. It is thus limited to problems with a small number of uncertainties. Note also that neglected dynamics cannot be handled [2].

- **mixed_mu_lb.** This routine is based on a power-algorithm, and thus computes a mixed $\mu$ lower-bound in polynomial-time. Note that a specific power algorithm was developed for this toolbox to handle skew uncertainties. When available, the power algorithm from the $\mu$ analysis and synthesis toolbox can also be used.

- **mixed_mu_lb_freq.** This higher level routine (based on the previous one) computes a $\mu$ or skew $\mu$ lower-bound on a frequency gridding.

- **mu_lb_with_freq.** This is also a high level routine. Here the aim is not to compute a lower-bound as a function of frequency but to compute a lower bound of the maximal value of $\mu$ over a frequency range. Different techniques [8] (combining frequency-domain and state-space methods) can be used to perform such a computation. Note here that no skew uncertainty is allowed. However, some performance problems can be tackled by considering $\Omega$ stability.

3.2.3 Delay magin

- **worst_case_margin.** Based on the small-gain theorem, this routine computes an estimate of the worst-case MIMO gain, phase and delay
margins when model uncertainties are maintained inside the unit ball.

3.3 Applicative examples

The toolbox contains 5 complementary applications, from the simplest one (De Gaston and Safonov’s example [3]) to the more complicated (Telescope mock-up example [5]). In this last application, the structured model perturbation contains 20 non-repeated parametric uncertainties, and the order of the system, with numerous flexible mode is 70.

In this paper, we focus on two applications (missile example and flexible aircraft) which are detailed in the following sections.

4 Missile example

4.1 Problem description

As a first illustration of this Toolbox, we consider a missile control problem along the longitudinal axis. The model which is used here is nonlinear and extracted from [13]. The main nonlinearities are the variation of the aerodynamic coefficients as a function of \( \alpha \), since these quantities are third order polynomials of \( \alpha \), whose range of variation is \([0, 20 \text{deg}]\). The actuator is modelled as a second order transfer function.

Since we are interested by local stability and performance properties of the missile around a nominal point, the nonlinear model is linearized at a medium value \( \alpha = 10 \text{ deg} \) of the angle of attack. The following LTI model is then obtained:

\[
\begin{align*}
\dot{\alpha} & = q + \frac{Q S}{\text{Mass} \ast V}(Z_\alpha \alpha + Z_\delta \delta) \\
\dot{q} & = \frac{Q S d}{I_y}(M_\alpha \alpha + M_\delta \delta) \\
\eta & = \frac{Q S}{\text{Mass} \ast g}(Z_\alpha \alpha + Z_\delta \delta)
\end{align*}
\]

(13)

This model is essentially parametrized by the 4 stability derivatives \( M_\alpha, M_\delta, Z_\alpha \text{ and } Z_\delta \). The issue in the following will be to analyse the local stability and performance properties of the closed loop missile in the presence of parametric uncertainties in these 4 stability derivatives and in the face of neglected dynamics, namely a high frequency bending mode. To this aim, the
parametrically uncertain missile model will be first transformed into a stan-
dard LFT structure $\mathcal{F}(H(s), \Delta_1)$, where $\Delta_1$ gathers the uncertainties in the
stability derivatives. On the other hand, the bending mode is represented by
an additive model perturbation $\Delta_2(s)$ and its template $1/W(s)$ is extracted
from [13]. The uncertain closed loop missile is presented in figure 4, where
$K(s)$ represents the missile autopilot. Note that the nominal uncertainties
in the stability derivatives are chosen as 5%. In the context of $H_\infty$ control,
the frequency domain performance is finally defined through the sensitivity
function $S$, i.e. the transfer between the commanded acceleration $\eta_c$ and the
tracking error $\epsilon = \eta_c - \eta$.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure4}
\caption{Closed loop missile system with uncertainties.}
\end{figure}

4.2 Standard $\mu$ problems

Standard $\mu$ analysis problems will be considered first, without neglected dy-
namics ($\Delta_2 = 0$) and without constraints on frequency domain performance.
The issue is then to solve a stability robustness analysis problem with 4
non-repeated real uncertainties.
4.2.1 Robust stability in the left half plane

Since the number of uncertainties is low, exponential-time techniques can be applied here. The two routines mu_zd and mu_dailey have thus been called (via mu_frequency_gridding) to compute a $\mu$ upper-bound and $\mu$ lower-bound respectively. The results are plotted on figure 5. Both plots are very close, which indicates that a reliable approximation of $\mu$ has been obtained on the considered frequency range. We then compute the mixed-$\mu$ upper-bound (using mu_ub). As can be observed on figure 5 both upper bounds are close, except at low frequencies. Noting moreover that both $\mu$ upper bounds are equal to 0.125 at the zero frequency. It can be concluded that $\mu$ is discontinuous at this frequency.

![Figure 5: $\mu$ for the missile example (robust stability inside the LHP).](image)

From the above analysis, it finally turns out that the maximal value of $\mu$ is 0.125 at the zero frequency. Robust stability can then be guaranteed for $\pm 40\%$ of uncertainty in the stability derivatives (i.e. the physical parameters of the missile model). Parametric uncertainties were indeed normalised by a weighting factor 0.05, so that the robustness margin is $\frac{0.05}{0.125}$. 
4.2.2 Robust stability in a truncated sector

As mentioned in section 3, the above classical $\mu$ tools can be applied to evaluate the robustness of a pole placement in a sector ($\Omega$ stability). Keeping the same frequency gridding as before, the new upper-bound (still obtained by the exponential-time technique implemented in $\text{muzd}$) is plotted on figure 6. The maximal value (obtained around $13\text{rad/s}$) is now 0.25. We thus may conclude that the pole placement remains inside the specified sector for $\pm 40\%$ of uncertainty in the stability derivatives.

![Figure 6: $\mu$ for the missile example (robust stability inside a truncated sector - initial gridding).](image)

To check this result, we further compute a $\mu$ lower-bound. Our objective here is to evaluate a lower-bound of the maximal value of $\mu$. The best routine to be used is then $\text{mu_lb_with_freq}$, which computes the lower-bound for some potentially critical frequencies only. The results are visualized on figure 6 by stars. Some problems clearly appear around $20\text{rad/s}$ and $80\text{rad/s}$ where the lower-bound seems to be higher than the computed upper-bound!

This suggests that the frequency-gridding for the upper-bound computation needs to be refined at least around critical frequencies. The corresponding results (plotted on figure 7) reveal that the peak around $80\text{rad/s}$ in the
previous analysis had been significantly under-evaluated. Yet, the robustness analysis was valid, since the highest peak was correctly detected.

![Figure 7: $\mu$ for the missile example (robust stability inside a truncated sector - refined gridding).](image)

### 4.3 Skew-$\mu$ problems

In this subsection we consider two analysis problems, which both require skew-$\mu$ computations.

#### 4.3.1 Robust stability with parametric uncertainties and neglected dynamics

Here, we are interested by a robust stability analysis in the LHP, in the presence of parametric uncertainties and neglected dynamics. Robust stability is first analysed with standard $\mu$ tools (see figure 8; the upper curve is the mixed $\mu$ upper bound, the lower one is the mixed $\mu$ lower bound). The maximal value of the upper bound is 0.236 at 216 rad/s, so that the maximal
uncertainty in the stability derivatives is $5/0.236 = 21.18\%$ (remember the weighting factor on the parametric uncertainties is 5\%). The result is nearly non-conservative, because of the very small gap between the $\mu$ lower and upper bounds at the critical frequency (lower and upper bounds nearly coincide at high frequencies).

![Figure 8: $\mu$ for the missile example (robust stability).](image)

However, the controller must tolerate a given amount of neglected dynamics, as defined by the template $W$ (see figure 4). It is thus logical to maintain this uncertainty inside its unit ball in our analysis problem. The maximal value of the mixed skew $\mu$ upper bound is obtained as 0.125 at the zero frequency, so that the maximal uncertainty in the stability derivatives becomes $5/0.125 = 40\%$ (see figure 9).
4.3.2 A robust performance problem

The issue is now to compute the maximal amount of model uncertainties (parametric uncertainties and neglected dynamics), for which the frequency-domain template on the sensitivity function $S(s)$ is still satisfied. The mixed skew $\mu$ upper bound is computed as a function of frequency and plotted on figure 4.3.2 (the lower curve corresponds to the skew mixed $\mu$ lower bound). The maximal value is obtained as 0.488 at 11.830 rad/s. The result is quite satisfactory, since the lower and upper bounds nearly coincide at this frequency. Robust performance is thus guaranteed for $\pm 10\%$ of uncertainties in the stability derivatives ($0.102 = \frac{0.05}{0.488}$).
Figure 10: skew $\mu$ for the missile example (robust performance).

Remark : If we had applied standard $\mu$ tools to solve this problem, it is interesting to point out that much more conservative results would have been obtained. In that case indeed, the maximum value of $\mu$ is close to 1, which means that robust performance is guaranteed for only $\pm 5\%$ of uncertainties in the stability derivatives.

5 Flexible aircraft example

5.1 Problem description

We first present the open-loop model of the flexible aircraft. A short description of the control law design is then proposed, and the robustness analysis problem is presented.
5.1.1 Open-loop model

A large civilian transport aircraft is considered. As shown in figure 11, a complete lateral model of the aircraft is obtained by adding its rigid and flexible models at the outputs.

The rigid model is characterized by two control inputs $\delta_r$ and $\delta_p$ (aileron and rudder deflection) and by four states $\beta$, $p$, $r$ and $\phi$ (sideslip angle, roll and yaw rates, roll angle). The linearized state-space equations, at a trim value ($\alpha_0, \theta_0$) of the angle of attack and of the pitch angle, are classically given as:

$$
\begin{align*}
\dot{\beta} &= Y_\beta \beta + (Y_p + \sin \alpha_0)p + (Y_r - \cos \alpha_0)r + \frac{g}{V} \phi + Y_{\delta p} \delta p + Y_{\delta r} \delta r \\
\dot{p} &= L_\beta \beta + L_p p + L_r r + L_{\delta p} \delta p + L_{\delta r} \delta r \\
\dot{r} &= N_\beta \beta + N_p p + N_r r + N_{\delta r} \delta r \\
\dot{\phi} &= p + \tan \theta_0 r
\end{align*}
$$

$g$ is the acceleration due to gravity, while $V$ is the aircraft speed. Note that the above model is characterized by its 14 stability derivatives $Y_\beta$, $Y_p$, $Y_r$, $Y_{\delta p}$, $Y_{\delta r}$, $L_\beta$, $L_p$, $L_r$, $L_{\delta p}$, $L_{\delta r}$, $N_\beta$, $N_p$, $N_r$ and $N_{\delta r}$.

On the other hand, the 12th order flexible model contains 6 bending modes (see table 1), whose damping ratios (resp. natural frequencies) evolve between 1.56% and 5.09% (resp. between 7.35 rad/s and 14.3 rad/s). See [5] for more details. Note as a final point that a second order actuator is added at the control input $\delta_p$, whereas a third order actuator is added at the control input $\delta_r$. 

Figure 11: Obtention of a complete model of the aircraft.
<table>
<thead>
<tr>
<th>damping ratio</th>
<th>natural frequency (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.56 \times 10^{-2} 14.3</td>
</tr>
<tr>
<td>2</td>
<td>2.16 \times 10^{-2} 13.5</td>
</tr>
<tr>
<td>3</td>
<td>2.42 \times 10^{-2} 12.5</td>
</tr>
<tr>
<td>4</td>
<td>3.29 \times 10^{-2} 7.35</td>
</tr>
<tr>
<td>5</td>
<td>5.07 \times 10^{-2} 14.1</td>
</tr>
<tr>
<td>6</td>
<td>5.09 \times 10^{-2} 8.62</td>
</tr>
</tbody>
</table>

Table 1: Characteristics of the bending modes

5.1.2 Controller design

An observed state feedback controller is synthesized for the flexible transport aircraft. The idea is simply to place the rigid closed loop poles to achieve performance and decoupling objectives, using a classical modal approach. On the other hand, some of the flexible modes (namely the critical ones for the performance) are shifted into the left half plane to increase their damping ratio. Note that only 4 out of the 6 bending modes have to be actively controlled.

5.1.3 Introduction of the uncertainties

As usual in real-world aeronautical applications, the above control law was designed on a nominal model. Yet, it must be emphasized that neither the rigid model of the aircraft, nor its flexible model are precisely known. Uncertainties consequently affect both stability derivatives and natural frequencies of the bending modes. As an example, the coefficient $Y_\beta$ should be rewritten as:

\[
Y_\beta = Y_\beta^0 (1 + x_1 \delta_1) \tag{15}
\]

$Y_\beta^0$ represents the nominal value of the coefficient. $\delta_1$, which is assumed to belong to the interval [-1,1], represents the normalized parametric uncertainty in $Y_\beta$. The constant scalar $x_1$ is finally used to weight the uncertainty in this coefficient $Y_\beta$, with respect to uncertainties in the other coefficients. This
scalar $x_1$ is called in the following "the nominal" uncertainty in the coefficient $Y_\beta$.

To apply robustness analysis, a standard interconnection structure $M(s) - \Delta$ must be derived. To achieve this, an LFT model of the open-loop aircraft has to be computed first. From the above discussion, a natural way to proceed consists in computing an LFT for the rigid part, and an LFT for the flexible part.

\[ \Delta = diag(\Delta_1, \Delta_2) \]

Because the uncertainties in the stability derivatives enter in an affine way in the state-space equations of the rigid model (see equations (14) and (15)), Morton’s method [12] can be applied to obtain an LFT model of the rigid aircraft, which contains 14 non repeated real scalars:

\[ \Delta_1 = diag(\delta_{r_1}, \delta_{r_2}, \ldots, \delta_{r_{14}}) \]
As for the flexible part, a rigorous approach would lead to an LFT model with 6 twice repeated real scalars [5]. Nevertheless, when assuming low values of the damping ratios and small variations of the natural frequencies, a first order approximation can be made, which enables to get a simpler model with 6 non repeated real scalars. For the ease of exposition, let us reduce the state space representation of the flexible model to the first bending mode. We obtain:

\[
\begin{pmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\end{pmatrix} = \begin{pmatrix} 1 & -2\xi_1 \omega_1 \\
0 & -\omega_1^2 \end{pmatrix} \begin{pmatrix} q_1 \\
q_2 \\
\end{pmatrix} + \begin{pmatrix} b_1 \\
b_2 \\
\end{pmatrix} u = A_1 \begin{pmatrix} q_1 \\
q_2 \\
\end{pmatrix} + \begin{pmatrix} b_1 \\
b_2 \\
\end{pmatrix} u
\]

Let \( \omega_1 = (1 + \delta_f_1) \omega_{01} \), where \( \delta_f_1 \) represents the normalized uncertainty in the frequency of the first bending mode. Under the assumptions above, \( A_1 \) can be approximated as:

\[
A_1 \approx \begin{pmatrix} 1 & -2\xi_1 \omega_1 \\
0 & -(1 + 2\delta_f_1) \omega_{01}^2 \end{pmatrix} = A_{0,1} + \Delta A_1
\]

where \( A_{0,1} \) represents the nominal value of matrix \( A_1 \), while the uncertainty matrix :

\[
\Delta A_1 = \begin{pmatrix} 0 & -2\delta_f_1 \omega_{01}^2 \\
0 & 0 \end{pmatrix}
\]

is obviously rank one. As a consequence, the parametric uncertainty \( \delta_f_1 \) will only appear once in the LFT model, and the structured model perturbation, which gathers the uncertainties in the frequencies of the 6 bending modes, is obtained as:

\[
\Delta_2 = \text{diag}(\delta_{f_1}, \delta_{f_2}, \ldots, \delta_{f_6})
\]

Finally, we obtain:

\[
\Delta = \text{diag}(\Delta_1, \Delta_2) = \text{diag}(\delta_{r_1}, \ldots, \delta_{r_14}, \delta_{f_1}, \ldots, \delta_{f_6})
\]

which thus contains 20 non-repeated real uncertainties on its diagonal.

Let \( K(s) \) the controller, which is to be connected to the LFT aircraft model \( F_l(H(s), \Delta) \). The standard interconnection structure \( M(s) - \Delta \) is then easily obtained, by noting that \( M(s) \) is the transfer matrix which is seen by the model perturbation \( \Delta \) in figure 13, i.e. the transfer matrix between \( w \) and \( z \).
5.2 Robust stability analysis

As explained in the previous subsection, uncertainties are simultaneously introduced in the 14 stability derivatives and in the natural frequencies of the 6 bending modes. The model perturbation then contains 20 real non-repeated scalars. The "nominal" uncertainties in the stability derivatives are chosen as 10%, while the "nominal" uncertainties in the frequencies are chosen as 20%.

On this challenging example the number of uncertainties is rather high and exponential-time methods cannot be applied. Furthermore, the system (order 46) contains six flexible modes so that the $\mu$ plot is expected to exhibit several high and narrow peaks. Consequently, classical techniques based on a pre-defined frequency gridding approach cannot be used. To compute a reliable mixed-$\mu$ upper-bound, we have then used the technique implemented in \texttt{mu\_max\_1}. Note that the method proposed in \texttt{mu\_max\_2} is not recommended here because of the large order of the system. An initial frequency gridding is provided with 50 points. This gridding is then automatically refined by the algorithm around critical frequencies. The results are presented in figure 14. Note that the height $\beta_i$ of each bar on a frequency interval $[\omega_i, \omega_{i+1}]$ is such that:

$$\forall \omega \in [\omega_i, \omega_{i+1}], \quad \mu(M(j\omega)) \leq \beta_i$$

The value of the upper bound of $\mu_{\text{max}}$ is found as 1.97 (between 9.93 and 10.00 rad/s).

To evaluate the conservatism of this result, several lower-bounds of the maximal value of $\mu$ have been computed using the routine \texttt{mu\_lb\_with\_freq}. 

Figure 13: LFT model of the closed-loop system.
Different options are available for this routine. Using the second option, we obtain $\mu_{\text{max}} = 1.966$ and the associated frequency is around 10 rad/s. This result is excellent since the gap between the $\mu$ bounds is less than 0.2%.

We have consequently proved that the flight control system can tolerate an uncertainty of 5.1% ($\approx 10/1.97$) in the stability derivatives and an uncertainty of 10.2% ($\approx 20/1.97$) in the frequencies of the bending modes.

To further illustrate the lower-bound and its associated worst-case perturbation $\Delta^*$, we have plotted on figure 15, the poles of the interconnection structure $M(s) - \alpha \Delta^*$ as the real parameter $\alpha$ evolves from 0 to 1. On the figure, the nominal values of the poles (without perturbation i.e. $\alpha = 0$) are represented by stars. The “final” values (associated to $\alpha = 1$) are represented by circles. We clearly note that the imaginary axis is reached around 10 rad/s which confirms the previous result. This plot is also interesting since it permits to identify the most critical pole and thus give ways to further improve the robustness of the controller.
Figure 14: Computation of $\mu$ as a function of $\omega$ for the flexible aircraft
Figure 15: Root locus associated to the destabilizing model perturbation
References


